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Cooperative location games based on the minimum diameter spanning Steiner subgraph problem

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ABSTRACT

In this paper we introduce and analyze new classes of cooperative games related to facility location models. The players are the customers (demand points) in the location problem and the characteristic value of a coalition is the cost of serving its members. Specifically, the cost in our games is the service diameter of the coalition.

We study the existence of core allocations for these games, focusing on network spaces, i.e., finite metric spaces induced by undirected graphs and positive edge lengths.

1. Introduction

The goal of cooperative game theory is to study ways to promote and enforce cooperation among agents (also called *players*) willing to cooperate [1,4,9,10,19]. A way to enforce cooperation is to find suitable allocations of the cost or benefit of such a cooperation among the players. These allocations must satisfy some rationality principles so that players are happy about their *payoffs*. Game theorists have analyzed the above problem over the years and have proposed several solutions, *core* allocations being the most universally accepted for the fairness properties they satisfy.

Basically, the core of a cooperative situation is the set of allocations of the total cost that satisfy the individual and collective *rationality principles*. In cost games, individual rationality means that no agent is going to be charged more than what he would pay acting by himself. Collective rationality ensures that no group of agents (also called *coalitions*) would be charged more than what they would pay when acting by themselves. The allocations satisfying those two principles can be considered *stable* in the sense that no agent or coalition would have incentives to break the *grand coalition* (the coalition consisting of all players), and thus cooperation is sustained. There is a large body of literature dealing with core concepts in cooperative game theory, e.g., [22].

Recall that a generic finite cooperative game is a pair (N, v) where N is the set of players and v is the characteristic function defined from 2^N to $\mathbb{R}$, which satisfies $v(\emptyset) = 0$, and assigns to each coalition $S \subseteq N$ a value (benefit or cost). For convenience, suppose that $N = \{u_1, \ldots, u_k\}$. With this notation, and assuming v is a cost function, the core of (N, v) is the set

$$C(N, v) = \{x \in \mathbb{R}^k : x(S) \le v(S),\ \forall S \subset N \text{ and } x(N) = v(N)\},$$

where $x(S) = \sum_{j:u_j \in S} x_j$, for all $S \subseteq N$.

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In the last decades there has been an increasing interest in studying cost allocation problems arising from and related to a variety of operations research problems and general optimization models (see [2,14]). Pioneering studies along this extensive line of research are the papers on assignment games, [27], linear production games, [21], network flow games, [13], and minimum spanning tree games, [4,1,9,10,19]. Even nowadays, this subject attracts a lot of interest among researchers, see e.g., the book by Nisan et al. [20] and the recent paper by Caprara and Letchford [3].

Our main interest is in cost allocation games related to location models. Some relevant references are [7,8,19,23,24,30].

The underlying optimization model of the games in this paper considers a connected undirected graph $G = (V, E)$ with positive edge lengths $\{l_e\}$, $e \in E$, where $V = \{v_0, v_1, \ldots, v_n\}$, and a set $N \subseteq V \setminus \{v_0\}$. Each edge in E is assumed to be rectifiable. We refer to interior points on an edge by their distances (along the edge) from the two nodes of the edge. $A(G)$ is the continuum set of points on the edges of G. For any pair of points $x, y \in A(G)$, we let $d(x, y)$ denote the length of a shortest path connecting x and y. We refer to $A(G)$ as the metric space induced by G and the edge lengths.

Also given is a finite subset of nodes $N \subseteq V \setminus \{v_0\}$. At times we refer to these nodes as *existing facilities*, or *demand points*. The distinguished node, v_0, is viewed as an essential element in the system, e.g., each demand point must have access to v_0. For motivation purposes, assume that the demand points represent customers or patients, and v_0 is the location of a repairman or a medical doctor who provides assistance or health services, respectively. Suppose first that the cost of serving a coalition $S \subseteq N$ is proportional to the length of the tour traveled by the server from his home base v_0, visiting each member of the coalition and returning to v_0. We then obtain the traveling salesman game, studied in [16,28].

In another situation v_0 can represent a central depot that all the existing communities must connect to. In this case the cost a coalition has to pay can be the length of a Steiner subtree connecting its members to v_0. This model is discussed in [9,10,19,29].

Our study is motivated by location models, where the time elapsed till the service is provided (response time) is critical. The cost function, also capturing the spreading of S and its distances from v_0, that we study in this paper is the diameter of the set $S \cup \{v_0\}$. As an example of this situation, consider the case in which a set of cities want to install a system to communicate among themselves. The cost of the communication system is proportional to the largest distance between a pair of cities, including the information center v_0.

We now formally define the two classes of cooperative cost games based on the above facility location problems, that we study in this paper.

For any subgraph $G' = (V', E')$ of $G = (V, E)$ we let $D^*(G')$ denote its diameter, i.e., the longest of the distances in the space $A(G')$ between all pairs of nodes in V':

$$D^*(G') = \max_{x,y \in V'} d_{G'}(x, y),$$

where $d_{G'}(x, y)$ denotes the shortest distance between x and y in $A(G')$. (If G' is not connected we define $D^*(G') = \infty$.) A pair of nodes, $v_i, v_j \in G'$ such that the distance between them in G' is equal to $D^*(G')$ is called a *diametrical pair* of G'.

Suppose that a coalition $S \subseteq N$ decides to use a subgraph $G' = (V', E')$, satisfying $S \cup \{v_0\} \subseteq V'$, to establish communication among its members, including v_0. Primary transmission points are established at the nodes in $S \cup \{v_0\}$. In many situations the communication cost may depend only on the distances in $A(G')$ between pairs of nodes in $S \cup \{v_0\}$, i.e., primary transmitters. However, in some situations, the technology used requires that, in addition, auxiliary transmission points are setup at all the extra nodes, $V' \setminus S \cup \{v_0\}$, in the subgraph. In these situations there is a cost associated with the inclusion of the extra nodes in the subgraph. The overall communication cost associated with the coalition S may then depend on the distances between all pairs of nodes in V', i.e., primary and auxiliary transmission points. The two games we consider in this paper refer to the two scenarios mentioned above, respectively.

The first game, denoted by (N, v_I), is the *minimum diameter location game* (MDLG), where for each coalition $S \subseteq N$, the cost is the maximum distance between pairs of nodes of $S \cup \{v_0\}$ in the selected subgraph G'. Since the additional nodes have no effect on the cost, in order to minimize its cost, the coalition will select $G' = G = (V, E)$. Thus, the characteristic function value is defined by the diameter of $S \cup \{v_0\}$ in $A(G)$, i.e.,

$$v_I(S) = \max_{x,y \in S \cup \{v_0\}} d(x, y).$$

Note that the above setup, defined only on a metric space $A(G)$, also captures the case where $N \cup \{v_0\}$ are points in a general metric space X. To model such a general case, consider the complete undirected graph $G^* = (N', E')$ with node set $N' = N \cup \{v_0\}$, and for each pair of nodes $x, y \in N'$ set the length of the edge connecting them in G^* to be equal to the distance between them in X.

The second minimum diameter situation introduced in this paper and denoted by (N, v_I^*), is called the *minimum Steiner subgraph diameter game* (MSSDG). In this game the cost of a coalition $S \subseteq N$, is the maximum distance between all pairs of nodes in the selected subgraph $G' = (V', E')$. The coalition will select the subgraph minimizing its cost. (Unlike the case of the first game, the best subgraph is not necessarily the entire graph G.) Formally, the characteristic function v_I^* is defined as follows:

For each subset $S \subseteq N$, define $\mathcal{G}(S)$ to be the set of all connected subgraphs of G spanning $S \cup \{v_0\}$. $\mathcal{G}(S)$ is called the set of *Steiner subgraphs* spanning $S \cup \{v_0\}$. Given a coalition $S \subseteq N$, we define its value, v_I^*, as the minimum diameter of a Steiner subgraph spanning $S \cup \{v_0\}$, i.e.,

$$v_I^*(S) = \min_{G' \in \mathcal{G}(S)} D^*(G').$$

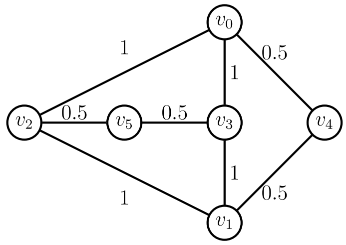

Fig. 1. The graph of Example 1.1.

A subgraph $G^*(S) \in \mathcal{G}(S)$ such that $v_I^*(S) = D^*(G^*(S))$ is called *S-optimal*. By definition, if $G^*(S) = (V', E')$ is S-optimal then we can assume without loss of generality that $G^*(S)$ is induced by its node set V', i.e., E' consists of all edges in G connecting pairs of nodes in V' only. In particular, if we let $E(S \cup \{v_0\})$ denote the set all edges of G connecting pairs of nodes in $S \cup \{v_0\}$, then, $G_{v_0}(S) = (S \cup \{v_0\}, E(S \cup \{v_0\}))$, *the subgraph induced by S and* v_0, is a subgraph of $G^*(S)$.

A game related to (N, v_I^*) is studied in [25]. In that game, called the minimum radius location game, or the minimum Steiner subtree diameter game, the value of a coalition S is the minimum diameter over all Steiner subtrees spanning $S \cup \{v_0\}$.

We emphasize the difference between $v_I(S)$ and $v_I^*(S)$. $v_I(S)$ is the longest of the distances in the space $A(G)$ between all pairs of nodes in $S \cup \{v_0\}$, while $v_I^*(S)$ is the longest of the distances in the space $A(G^*(S))$ between all pairs of nodes of an S-optimal Steiner subgraph $G^*(S)$. In particular,

$$v_I(S) \leq v_I^*(S).$$

Also, if the edge lengths of G satisfy the triangle inequality then $v_I(S) = v_I^*(S)$, for any $S \subseteq N$.

In the next example we illustrate the difference between the two characteristic functions.

Example 1.1. Consider the 6 node graph $G = (V, E)$ with $V = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ and $E = \{(v_0, v_2), (v_0, v_3), (v_0, v_4), (v_1, v_2), (v_1, v_3), (v_1, v_4), (v_2, v_5), (v_3, v_5)\}$. Let the length of the edges (v_0, v_4), (v_1, v_4), (v_2, v_5) and (v_3, v_5) be equal to 0.5, and let the length of the remaining 4 edges be equal to 1 (see Fig. 1).

Let $S = \{v_1, v_2, v_3\}$. Recall that in order to calculate $v_I(S)$, it is sufficient to consider the entire graph G, and calculate the shortest distances in $A(G)$ between all pairs of nodes in $S \cup \{v_0\}$. The maximum of all these distances is 1. Then $v_I(S) = 1$. To calculate $v_I^*(S)$ we need to consider 4 subgraphs, i.e., the subgraphs induced by the node sets $S \cup \{v_0\}$, $S \cup \{v_0, v_4\}$, $S \cup \{v_0, v_5\}$, and V. It is easy to check that the diameter of all these 4 subgraphs is 2. For example, the diameter of the entire graph is given by $d(v_4, v_5) = 2$. Therefore $v_I^*(S) = 2$.

Finding a minimum diameter spanning Steiner subgraph of a given subset of nodes seems to be an interesting combinatorial problem which, to the best of our knowledge, has not been discussed in the literature. (Note that unlike the minimum length spanning Steiner subgraph, the minimum diameter spanning subgraph is not necessarily a subtree.) We elaborate on the complexity of this problem in Sections 3.1.1 and 3.1.2.

Our goal is to investigate the existence of core elements for the two games. In Section 2 we show that $C(N, v_I)$ is always nonempty. Moreover, there is an extreme point of $C(N, v_I)$, which has at most two positive components (associated with a diametrical pair). We also prove that testing whether a given vector x is in $C(N, v_I)$ is NP-hard. In Section 3 we study the game (N, v_I^*). We show that its core $C(N, v_I^*)$ may be empty when the set of players is a proper subset of $V \setminus \{v_0\}$. On the other hand, if the set of players is equal to $V \setminus \{v_0\}$, then $C(N, v_I) \subseteq C(N, v_I^*)$. We also show that the problem of computing $v_I^*(S)$ for a given subset of players is NP-hard to approximate within a multiplicative factor strictly smaller than 4/3, and $v_I^*(S)$ can be efficiently approximated within a factor 2. Finally, we prove that for any coalition S, $v_I(S) \leq v_I^*(S) \leq 2v_I(S)$, which in turn implies that any vector in $C(N, v_I)$ is a 1/2-budget balanced allocation of the game $C(N, v_I^*)$. In Section 4, some results on the calculation of the nucleolus and the Shapley value are shown for the particular case of tree networks. We also present a compact formulation of the core in this case. The paper ends with some conclusions.

2. The minimum diameter location game, (N, v_I)

This section is devoted to the MDLG. We first prove that $C(N, v_I)$ is nonempty.

Theorem 2.1. *Given a graph $G = (V, E)$, and a subset $N \subseteq V \setminus \{v_0\}$, let (N, v_I) be the respective minimum diameter location game, defined over $A(G)$. Then, there is an extreme point of $C(N, v_I)$, which has at most two positive components.*

Proof. Let $v_i, v_j \in N \cup \{v_0\}$ such that $v_I(N) = d(v_i, v_j)$.

If $v_j = v_0$, define the allocation x' by setting $x'_i = v_I(N) = d(v_i, v_0)$, and $x'_k = 0$, for any $k \neq i$. It is easy to see that x' is in the core since for each coalition S such that $v_i \in S$, we have $x'(S) = x'_i = d(v_i, v_0) \leq v_I(S)$.

Next suppose that $v_i \neq v_0$ and $v_j \neq v_0$. We present two extreme points of $C(N, v_I)$. First, define the allocation x' by setting $x'_i = d(v_i, v_0)$, $x'_j = v_I(N) - d(v_i, v_0)$, and $x'_k = 0$ for any $k \neq i, j$. Note that by the triangle inequality, $x'_j \leq d(v_j, v_0) = v_I(\{v_j\})$.

Then, $x'(S) = v_I(N) = d(v_i, v_j) \leq v_I(S)$, for each coalition S, satisfying $v_i, v_j \in S$. Also, $x'(N) = v_I(N)$. If $v_i \in S$ and $v_j \notin S$, then $x'(S) = x'_i = d(v_i, v_0) \leq v_I(S)$. Similarly, if $v_j \in S$ and $v_i \notin S$, then $x'(S) = x'_j \leq d(v_j, v_0) \leq v_I(S)$.

A second extreme point of $C(N, v_I)$, x'', is similarly defined by setting, $x''_j = d(v_j, v_0)$, $x''_i = v_I(N) - d(v_j, v_0)$, and $x''_k = 0$ for any $k \neq i, j$. This concludes the proof. $\square$

In spite of the facts that $C(N, v_I)$ is nonempty and that $v_I(S)$ can efficiently be computed for any coalition S, we next show that testing membership in the core for a given vector x is NP-hard for general graphs. Note that the latter task amounts to testing whether $\min_{S\subseteq N}(v_I(S) - x(S)) \geq 0$.

Formally, given an MDLG with an underlying graph $G = (V, E)$ with positive edge weights, and an allocation vector x, the *core membership decision problem* is to determine whether x is not in the core $C(N, v_I)$.

Theorem 2.2. *The core membership decision problem is NP-hard even when $G = (V, E)$ is a complete graph, $N = V \setminus \{v_0\}$, the edge lengths satisfy the triangle inequality, and x distributes the total cost $v_I(N)$ equally.*

Proof. We formulate the independent set problem [6] as an instance of the core membership decision problem. An instance of the NP-Complete independent set problem is an undirected graph $G_1 = (V_1, E_1)$ and an integer k, and the decision problem is whether G_1 has an independent set (i.e., a set of nodes such that no pair of them are adjacent) of size greater than k. Without loss of generality we may assume that $|V_1|$ is even and $k = |V_1|/2$. (If $k \leq |V_1|/2$, add $|V_1| - 2k$ isolated nodes to G_1. If $k > |V_1|/2$, add a clique with $2k - |V_1|$ nodes to G_1.)

Let $G_1 = (V_1, E_1)$ be an undirected graph with $V_1 = \{v_1, \ldots, v_n\}$. Let $G_2 = (V_1, E_2)$ be the complete graph with node set V_1. Associate a positive length with each edge of E_2 as follows: If $e \in E_1$ then set the length of e to be equal to n. If $e \notin E_1$ then set the length of e to be equal to $n/2$. Let $G_3 = (V_1 \cup \{v_0\}, E_3)$ be the graph obtained from G_2 by adding the node v_0 and the n edges connecting v_0 to the n nodes in V_1. The length of each one of these n edges is set to be equal to $n/2$. Note that G_3 is a complete graph with $n + 1$ nodes, and its edges satisfy the triangle inequality.

Next, set $N = V_1$ and consider the game (N, v_I), defined on $A(G_3)$. In order to prove our claim, we will show that $x = (1, \ldots, 1)$ is not in $C(N, v_I)$ if and only if the graph G_1 has an independent set of cardinality greater than $n/2$. We assume without loss of generality that E_1 is nonempty, and therefore $v_I(N) = n$.

First note that $v_I(S) \in \{n, n/2\}$ for any $S \subseteq N$. Also, $v_I(N) = n = \sum_{j=1}^{n} x_j$.

Suppose that G_1 has an independent set S with $|S| > n/2$. Then, by definition $v_I(S) = n/2 < |S| = \sum_{v_j\in S} x_j = x(S)$, and therefore $x \notin C(N, v_I)$.

Next suppose that there is a subset $S \subseteq N$ such that $v_I(S) < x(S) = \sum_{v_j\in S} x_j = |S| \leq n$. Therefore, $v_I(S) = n/2$, and $|S| > n/2$. In particular, the subgraph induced by S has its diameter equal to $n/2$. By the definition of the edge lengths, S is an independent set of G_1 (otherwise there would exist a pair $v_i, v_j \in S$ with $d(v_i, v_j) = n$). Since $|S| > n/2$, the result is proven. $\square$

In view of the above result it is unlikely that there is a formulation of $C(N, v_I)$ involving only a polynomial number of linear constraints. In Section 4 we present an efficient representation of $C(N, v_I)$ for tree graphs.

3. The minimum Steiner subgraph diameter location game, (N, v_I^*)

Unlike the game (N, v_I), we will show that the core of the game (N, v_I^*) can be empty when N is a proper subset of $V \setminus \{v_0\}$, and it is nonempty when $N = V \setminus \{v_0\}$. In the latter case we call the game *complete*. Note that when the game is complete, $v_I(N) = v_I^*(N)$. This is summarized in the following result.

Proposition 3.1. *Let $N \subseteq V \setminus \{v_0\}$. Then for any $S \subseteq N$, $v_I(S) \leq v_I^*(S)$. Moreover, if $N = V \setminus \{v_0\}$, then $v_I(N) = v_I^*(N)$.*

Theorem 3.1. *Given a graph $G = (V, E)$, suppose that $N = V \setminus \{v_0\}$. Let (N, v_{I^*}) be the respective minimum Steiner subgraph diameter location game, defined over $A(G)$. Then, there is an extreme point of $C(N, v_I^*)$, which has at most two positive components.*

Proof. The result follows from the above proposition and Theorem 2.1, since $C(N, v_I) \subseteq C(N, v_I^*)$ in this case. $\square$

The next result follows directly from Theorem 2.2 since the games (N, v_I) and (N, v_I^*) are identical when the underlying graph is complete and its edges satisfy the triangle inequality.

Theorem 3.2. *Let $G = (V, E)$ be a complete graph and let $N = V \setminus \{v_0\}$. Suppose that its edge lengths satisfy the triangle inequality. Let $x = (1, 1, \ldots, 1)$. Then the problem of determining whether x is not in $C(N, v_I^*)$ is NP-hard.*

The next example shows that the core of the game (N, v_I^*) might be empty.

Example 3.1. Consider the graph $G = (V, E)$, where

$$V = \{v_0, v_1, v_2, v_3, v_4, v'_1, v'_2, v'_3, v'_4\}$$

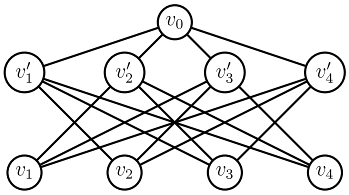

Fig. 2. The graph of Example 3.1.

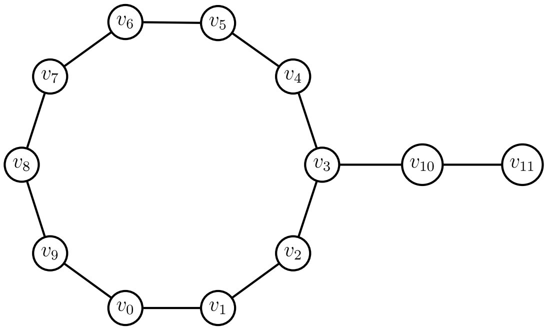

Fig. 3. The graph of Example 3.2.

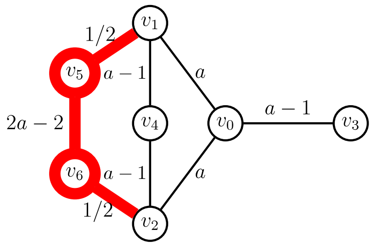

Fig. 4. Examples 3.3 and 3.4 in thin and thick lines, respectively.

and

$$E = \{(v_0, v'_1), (v_0, v'_2), (v_0, v'_3), (v_0, v'_4), (v_1, v'_2), (v_1, v'_3), (v_1, v'_4), (v_2, v'_1), (v_2, v'_3), (v_2, v'_4), (v_3, v'_1), (v_3, v'_2), (v_3, v'_4), (v_4, v'_1), (v_4, v'_2), (v_4, v'_3)\}.$$

All edges are of unit length, see Fig. 2.

Consider the case where the set of players is $N = \{v_1, v_2, v_3, v_4\}$. It is easy to see that for each S with $|S| = 3$, $v^*_I(S) = 2$, and $v^*_I(N) = 3$. (The superset yielding $v^*_I(N)$ must include some node v'_i and the distance between v_i and v'_i on the entire graph is 3.) If x was in the core it would have to satisfy, $x_1+x_2+x_3 \le 2$, $x_1+x_3+x_4 \le 2$, $x_1+x_2+x_4 \le 2$, and $x_2+x_3+x_4 \le 2$. Summing these inequalities yields $3(x_1 + x_2 + x_3 + x_4) \le 8$, which is not possible when $x_1 + x_2 + x_3 + x_4 = 3$.

Also note that $v_I(N) = 2 < v^*_I(N)$.

In the above example the set of players N is a proper subset of $V \setminus \{v_0\}$, and therefore it does not contradict Proposition 3.1 nor Theorem 3.1.

The results on the emptiness and nonemptiness of $C(N, v^*_I)$ have also been observed in some other combinatorial optimization games. For example, in the minimum spanning tree game, $v(S)$ is the total length of a Steiner subtree spanning $S \cup \{v_0\}$. The core of this game is nonempty if the set of players satisfies $N = V \setminus \{v_0\}$, [9], although it can be empty if N is a proper subset of $V \setminus \{v_0\}$, [29].

Remark 3.1. Theorem 3.1 implies that some variants of the complete version of (N, v^*_I) also have nonempty cores when $N = V \setminus \{v_0\}$. Consider for example the game (N, v'_I), defined by the characteristic function

$$v'_I(S) = D^*(G_{v_0}(S)).$$

Unlike the games (N, v_I) and (N, v^*_I), this game is not even monotone, and therefore can have core allocations which are not nonnegative. Nevertheless, since $v^*_I(S) \le v'_I(S)$, for all coalitions $S \subseteq N$, and $v^*_I(N) = v'_I(N)$, we conclude that $C(N, v^*_I) \subseteq C(N, v'_I)$. In fact, it is easy to see that $C(N, v^*_I) = C(N, v'_I) \cap \mathbb{R}^N_+$.

3.1. Computing $v_I^*(S)$

In this section we show several examples and observations on properties and approximability of $v_I^*(S)$.

Remark 3.2. As noted in the Introduction, for any S there is a minimum diameter Steiner subgraph of S which contains $G_{v_0}(S)$. Clearly, the entire graph may not be S-optimal for some S. (Consider, for example, a 2-star tree with $V = \{v_0, v_1, v_2\}$ and positive edge lengths centered at v_0, $N = \{v_1, v_2\}$ and $S = \{v_1\}$.)

Moreover, even if the diameter of $G_{v_0}(S)$ is unique and strictly greater than the distance in G between the unique diametrical pair of $G_{v_0}(S)$, the minimum diameter Steiner subgraph of S can still be $G_{v_0}(S)$. In other words, adding to $G_{v_0}(S)$ a shortest path in G between the diametrical pair may increase the diameter, as shown in the following example.

Example 3.2. Consider the following 12-node graph G with unit edge lengths. There is a 10-node cycle where the nodes v_i, $i = 0, 1, \ldots, 9$, are cyclically ordered. In addition there are the edges (v_3, v_{10}) and (v_{10}, v_{11}), see Fig. 3.

Let $S = \{v_1, \ldots, v_6, v_{10}, v_{11}\}$. The diameter of $G_{v_0}(S)$ is equal to 6, and it is uniquely attained by the pair v_0, v_6. The shortest distance in G between v_0 and v_6 is 4. However, adding the shortest path in G between v_0 and v_6 to $G_{v_0}(S)$ yields the graph G itself with $D^*(G) = 7$.

The above example can be extended to show that adding to the graph a shortest path between a diametrical pair of $G_{v_0}(S)$ can asymptotically increase the diameter by a factor of 1/2.

Example 3.3. Consider the 5 node graph $G = (V, E)$, where $V = \{v_0, v_1, \ldots, v_4\}$ and $E = \{(v_0, v_1), (v_0, v_2), (v_0, v_3), (v_1, v_4), (v_2, v_4)\}$, depicted in Fig. 4 (thin line). For $a \in \mathbb{Z}$, the lengths of its edges are: a for edges (v_0, v_1) and (v_0, v_2), and $a - 1$ for the other 3 edges, (v_0, v_3), (v_1, v_4) and (v_2, v_4).

Let $S = \{v_1, v_2, v_3\}$. Then, $G_{v_0}(S)$ is the 3-star centered at node v_0. Adding the shortest path between the diametrical pair $\{v_1, v_2\}$ will increase the diameter from $2a$ to $3a - 2$.

Remark 3.3. Note, however, that an increase by a factor of 1/2, observed in the above example, is the worst case over all graphs.

Consider a general undirected connected graph $G = (V, E)$. Let $S \subseteq N$. Suppose without loss of generality that v_1, v_2 is a diametrical pair in $G_{v_0}(S)$. Let $P(v_1, v_2)$ be a shortest path in $A(G)$ between them. Let $G'(S)$ denote the graph obtained from $G_{v_0}(S)$ by adding $P(v_1, v_2)$. Then, for each pair of nodes, $v_i \in G_{v_0}(S)$ and $v_j \in P(v_1, v_2)$,

$$\begin{aligned}
d_{G'(S)}(v_i, v_j) &\le \min(d_{G_{v_0}(S)}(v_i, v_1) + d(v_1, v_j); d_{G_{v_0}(S)}(v_i, v_2) + d(v_2, v_j)) \\
&\le d_{G_{v_0}(S)}(v_1, v_2) + \min(d(v_1, v_j); d(v_2, v_j)) \\
&\le d_{G_{v_0}(S)}(v_1, v_2) + (1/2)d(v_1, v_2) \\
&\le (3/2)d_{G_{v_0}(S)}(v_1, v_2).
\end{aligned}$$

(For each subgraph G^*, and a pair of nodes v_s, v_t, we let $d_{G^*}(v_s, v_t)$ denote the distance between v_s and v_t in $A(G^*)$.)

Also, note that even when the addition of a shortest path does not improve the diameter, it is still possible to improve it by adding some other path, as shown in the next example.

Example 3.4. Consider the graph in Example 3.3. Instead of adding the edges $(v_1, v_4), (v_4, v_2)$, add the edges $(v_1, v_5), (v_5, v_6), (v_6, v_2)$ of lengths $1/2$, $2a - 2$, and $1/2$, respectively, see Fig. 4.

The length of the path that we have added is $2a - 1$, which is larger than the length of the path (v_1, v_4, v_2). Nevertheless, its addition to the 3-star will decrease the diameter from $2a$ to $2a - 1/2$.

3.1.1. A 2-approximation for $v_I^*(S)$

A modification of the above construction can be used to obtain a 2-approximation for $v_I^*(S)$, and prove that $v_I^*(S) \le 2v_I(S)$.

Consider the game (N, v_I^*), defined on a general connected undirected graph G. Let $S \subseteq N$. To approximate $v_I^*(S)$ consider the subgraph $G_{v_0}(S)$. For each pair of nodes v_i, v_j of $G_{v_0}(S)$ add to $G_{v_0}(S)$ a shortest path, say $P(v_i, v_j)$, connecting the pair in $A(G)$. Let L^* denote the maximum length of these paths. By definition $L^* = v_I(S)$. Let $G'(S)$ denote the graph obtained after the addition. It is easy to see that for each pair v_t, v_s in $G'(S)$, $d_{G'(S)}(v_t, v_s)$, the distance between them in $A(G'(S))$, satisfies $d_{G'(S)}(v_t, v_s) \le 2L^* = 2v_I(S)$. Thus,

$$v_I(S) \le v_I^*(S) \le D^*(G'(S)) \le 2v_I(S).$$

We conclude that $D^*(G'(S))$ is a 2-approximation of $v_I^*(S)$, and

Theorem 3.3. *Given an undirected graph $G = (V, E)$, suppose that $N \subseteq V \setminus \{v_0\}$. Then for any $S \subseteq N$, $v_I(S) \le v_I^*(S) \le 2v_I(S)$.*

The following example shows that the factor 2 is asymptotically best possible for the approximation $D^*(G'(S))$.

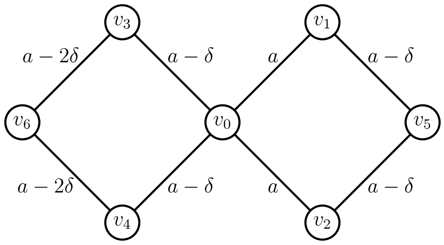

Fig. 5. The graph of Example 3.5.

Example 3.5. Consider a 4-star centered at v_0. The nodes v_1, v_2 are connected to v_0 with edges of length a, and the nodes v_3, v_4 are connected to v_0 with edges of length $a-\delta$. Define $S = \{v_1, v_2, v_3, v_4\}$. Next, add node v_5 to the star and connect it to nodes v_1 and v_2 with edges of length $a-\delta$. Also, add node v_6 to the star and connect it to nodes v_3 and v_4 with edges of length $a-2\delta$, see Fig. 5. We then have $v_I^*(S) = 2a$ and $D^*(G'(S)) = d(v_5, v_6) = 4a - 4\delta$. Hence, $D^*(G'(S)) = 2v_I^*(S) - 4\delta$.

Example 1.1 shows that $2v_I(S)$ is a tight upper bound on $v_I^*(S)$.

Using game theory terminology, [3], the last theorem implies that every vector $x \in C(N, v_I)$, is a 1/2-budget balanced vector of the game $C(N, v_I^*)$, i.e., for any $S \subseteq N$, $x(S) \leq v_I^*(S)$, and $(1/2)v_I^*(N) \leq x(N) \leq v_I^*(N)$.

3.1.2. Inapproximability of $v_I^*(S)$

Generally, the problem of computing $v_I^*(S)$ for a given coalition is NP-hard, [18]. It is not known whether the approximation factor 2 is best possible, although to get a better approximation a different solution approach would be required. However, we have slightly modified the NP-hardness proof of Levin [18] to show that even approximating within a constant factor α, $\alpha < 4/3$, is already NP-hard. Since Levin's proof is unpublished, for the sake of completeness, we include a proof of our modified inapproximability result.

Lemma 3.1. *For any $\alpha < 4/3$, approximating $v_I^*(S)$ within a constant factor α, is NP-hard.*

Proof. The reduction is from SAT.

Consider a SAT instance whose literals are $w_1, \ldots, w_n$, and its clauses are $C_1, \ldots, C_m$. Let us denote the negation of w_i by u_i. Construct a graph whose node set is $w_1, \ldots, w_n, u_1, \ldots, u_n, C_1, \ldots, C_m, t$ (i.e., one node for each literal or its negation, one node for each clause and one additional node for the true assignment). The set S is defined by $S = \{C_1, \ldots, C_m\}$ and $v_0 = t$. It remains to define the edge lengths.

Let $0 < \varepsilon \leq 1/3$. Each clause is connected to its literals via edges of length $1-\varepsilon$. The length of each edge connecting two literals is $1+\varepsilon$, if they correspond to different variables, and for every $i = 1, \ldots, n$, the length of the edge (w_i, u_i) is $2+2\varepsilon$. The length of an edge between any two clauses is 2. The length of an edge between t and a clause node is 3. Finally, the length of an edge between t and w_i or u_i (for every $i = 1, \ldots, n$) is $1+\varepsilon$.

In this graph there is a superset S', $S \subseteq S'$, such that the subgraph induced by $S' \cup \{v_0\}$ has diameter at most 2 if and only if the SAT formula can be satisfied.

First note that if there is a satisfying assignment then picking the true literals with S gives the correct S' with diameter at most 2.

It remains to consider the other direction. Assume that there is a superset S' such that the induced diameter is at most 2. Note that by the constraint $0 < \varepsilon \leq 1/3$, for every $i = 1, \ldots, n$, S' may contain either w_i or u_i but not both, because the distance between these two nodes is greater than 2. (By the choice of ε this distance is equal to $2+2\varepsilon$.) Assign a true value to the node that belongs to S' among the two nodes.

Then, note that for every C_j there is a literal whose node is in S' and therefore every clause has a true literal, so this assignment satisfies the SAT formula.

To observe that approximating within a constant factor $\alpha < 4/3$ is NP-hard we note that in the above construction, if $v_I^*(S) > 2$ then $v_I^*(S) = 2+2\varepsilon$. Thus, choosing $\varepsilon = 1/3$ yields the result. $\square$

4. Tree networks

In this section we focus on the interesting case of tree graphs. Let $T = (V, E)$ be a tree graph with $V = \{v_0, v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_n\}$. Let $N \subseteq V \setminus \{v_0\}$ be the set of players. It is easy to see that in this case the two games, (N, v_I) and (N, v_I^*), coincide, i.e., $v_I(S) = v_I^*(S)$, for any $S \subseteq N$. We present an $O(n^3)$ algorithm for calculating the Shapley value. In addition, we provide a compact representation of the core of the game, which has $O(n^2)$ linear constraints.

First, it is shown in [31] that the diameter function is submodular, i.e., for each pair of subsets $S_1 \subseteq N$, $S_2 \subseteq N$,

$$v_I(S_1 \cup S_2) + v_I(S_1 \cap S_2) \leq v_I(S_1) + v_I(S_2).$$

As a result we conclude that the minimum diameter game on a tree network is concave. (See [26] for a characterization of the core of concave games.)

Also, since the game is concave, its nucleolus [15] can be computed in polynomial time, (see [5,17]), and membership in the core can be verified in polynomial time.

Moreover, since the diameter game (N, v_I) is concave, it follows that its Shapley value is always an allocation in the core of the game. Recall that the Shapley value is the allocation $\phi = (\phi_1, \ldots, \phi_n)$ given by

$$\phi_k = \sum_{S \subseteq N \setminus \{v_k\}} \frac{s!(n-s-1)!}{n!} (v_I(S \cup \{v_k\}) - v_I(S)) \quad \forall\, v_k \in N, \tag{1}$$

where $s = |S|$. (For convenience we assume that $|N| = n$.)

Generally, assuming that the characteristic function is already known, it might take an exponential number of basic operations with respect to the number of players, to explicitly calculate ϕ by the above expression. In the rest of the section we show that, for diameter games defined on tree graphs, ϕ can be calculated in polynomial time.

First note that for each possible value of $v_I(S \cup \{v_k\}) - v_I(S)$ there can be several combinations of coalitions S and players v_k giving this value.

Consider some $v_k \in N$, and a coalition $S \subseteq N \setminus \{v_k\}$. In order to analyze the values that $v_I(S \cup \{v_k\}) - v_I(S)$ can take on, we use the classical result of [11]. Given a subtree T', to find a diametrical pair of nodes of $T' = (V', E')$, arbitrarily choose some node of T', say v_p. Let v_q satisfy $d(v_q, v_p) = \max_{v_i \in V'} d(v_i, v_p)$, and let v_r satisfy $d(v_r, v_q) = \max_{v_i \in V'} d(v_i, v_q)$. The pair $\{v_q, v_r\}$ is a diametrical pair of T'. This pair can therefore be found in $O(|V'|)$ time. This result implies the following property.

Lemma 4.1. *Let $\{v_q, v_r\}$ be a diametrical pair of the tree $T' = (V', E')$, and let $T'' = (V'', E'')$ be a subtree of T' such that $v_q \in V''$. Then, there is a node $v_s \in V''$, such that $\{v_q, v_s\}$ is a diametrical pair of T''.*

We now apply the lemma to the case where T' is the minimal subtree spanning $S \cup \{v_0, v_k\}$ and T'' is the minimal subtree spanning $S \cup \{v_0\}$.

The following cases may arise:

- $S = \emptyset$. Then, $v_I(S \cup \{v_k\}) = d(v_k, v_0)$, $v_I(S) = 0$.
- $v_I(S \cup \{v_k\}) = d(v_0, v_k)$. Then, there exists $v_j \in S$ such that $v_I(S) = d(v_j, v_0)$.
- $v_I(S \cup \{v_k\}) = d(v_k, v_j)$, for some $v_j \in S$. In this case two subcases are possible:
  1. $v_I(S) = d(v_j, v_0)$,
  2. $v_I(S) = d(v_j, v_t)$, for some $v_t \in S$.
- $v_I(S \cup \{v_k\}) = d(v_j, v_i)$, for some pair $v_j, v_i \in S \cup \{v_0\}$. Then, $v_I(S) = d(v_j, v_i)$ and therefore $v_I(S \cup \{v_k\}) - v_I(S) = 0$. (We do not have to take this case into consideration in order to calculate the Shapley value.)

Note that the implications stated with respect to the above cases follow from Lemma 4.1, since if $v_I(S \cup \{v_k\}) = d(v_k, v_j)$, $v_j \in S \cup \{v_0\}$, then $v_I(S)$ is given by the distance from v_j to another point of $S \cup \{v_0\}$.

Using the above properties, the following algorithm to calculate the Shapley value is proposed.

4.1. Algorithm: computing the Shapley value

For each coalition S, the value $v_I(S)$ is a continuous function of the edge lengths of the tree. Therefore, the Shapley value is continuous in the edge lengths. Hence, by perturbing the edge lengths, if necessary, we may assume without loss of generality that the distances between the nodes of the tree are distinct. (Specifically, if the edge set of the tree T is given by $E = \{e_1, \ldots, e_n\}$, then for each edge e_j, we add the term ε^j to its length $l(e_j)$.)

The algorithm we propose calculates the possible marginal values $(v_I(S \cup \{v_k\}) - v_I(S))$ by finding the values of the diameters of subsets of nodes. These diameters are determined by all possible pairs of nodes in V.

In the first phase of the algorithm $v_I(\{v_i, v_j\})$ is calculated for each pair of nodes $v_i, v_j \in N$. (v_i and v_j are not necessarily distinct.) The effort of this step is $O(n^2)$.

In the second phase we consider all pairs of nodes in N.

- Consider first a pair of distinct nodes $v_i, v_j \in N$, such that $v_I(\{v_i, v_j\}) = d(v_i, v_j)$. By the above nondegeneracy assumption we have $d(v_i, v_j) > d(v_i, v_0)$ and $d(v_i, v_j) > d(v_j, v_0)$.

  Let $T(i, j)$ be the maximal subtree with the diameter value equal to $d(v_i, v_j)$. It clearly takes $O(n)$ time to calculate $T(i, j)$. (Note that if x is the midpoint of the unique path connecting v_i with v_j, then the node set of $T(i, j)$ is given by $\{v_t : d(v_t, x) \le d(v_i, v_j)/2\}$.)

  Let $N(i, j)$ be the number of nodes in $T(i, j) \setminus \{v_i, v_j\}$.

  If v_k is a node in $T(i, j)$, then for each coalition $S \subseteq T(i, j)$, containing both v_i and v_j, we have $v_I(S \cup \{v_k\}) - v_I(S) = d(v_i, v_j) - d(v_i, v_j) = 0$. Thus, it is sufficient to consider only the case where $v_k \notin T(i, j)$. Note that in this case, by the maximality property of $T(i, j)$, we have $v_I(S \cup \{v_k\}) = \max(d(v_k, v_i), d(v_k, v_j))$. Hence, in this case for each coalition $S \subseteq T(i, j)$, $v_k \notin S$, containing both v_i and v_j, we have $v_I(S \cup \{v_k\}) - v_I(S) = \max(d(v_k, v_i), d(v_k, v_j)) - d(v_i, v_j)$.

  For each $v_k \in N$, define

  $$A_k = \{\{v_i, v_j\} : v_I(\{v_i, v_j\}) = d(v_i, v_j), \max(d(v_k, v_i), d(v_k, v_j)) > d(v_i, v_j)\}.$$

- Next we consider the case where $v_I(\{v_i, v_j\}) > d(v_i, v_j)$. Assume without loss of generality that $v_I(\{v_i, v_j\}) = d(v_i, v_0)$. Let $T(i, 0)$ be the maximal subtree with the diameter value equal to $d(v_i, v_0)$. Let $N(i, 0)$ be the number of nodes in $T(i, 0) \setminus \{v_i\}$. As above, it is sufficient to consider only the case where $v_k \notin T(i, 0)$. Note that in this case we have $v_I(S \cup \{v_k\}) = \max(d(v_k, v_i), d(v_k, v_0))$. Thus, in this case for each coalition $S \subseteq T(i, 0)$, containing v_i, we have $v_I(S \cup \{v_k\}) - v_I(S) = \max(d(v_k, v_i), d(v_k, v_0)) - d(v_i, v_0)$.

  For each $v_k \in N$, define

$$B_k = \{v_i : \max(d(v_k, v_0), d(v_k, v_i)) > d(v_i, v_0)\}.$$

Consider now a subtree $T(i, j)$, with $i, j > 0$. Then in this case, the number of times that the triplet $\{v_i, v_k, v_j\}$ and the pair $\{v_i, v_j\}$ assume the marginal value $\max(d(v_k, v_i), d(v_k, v_j)) - d(v_i, v_j)$, for coalitions of size $r + 2$, $r = 0, \ldots, N(i, j)$ (r different nodes plus the two nodes v_i, v_j) is $\binom{N(i,j)}{r}$. Similarly, for a subtree $T(i, 0)$ the number of times that the pair $\{v_i, v_k\}$ and the singleton $\{v_i\}$ assume the marginal value $\max(d(v_k, v_0), d(v_k, v_i)) - d(v_i, v_0)$, for coalitions of size $r + 1$, $r = 0, \ldots, N(i, 0)$ (r different nodes plus v_i) is $\binom{N(i,0)}{r}$.

Therefore, for each pair $v_i \in N$ and $v_j \in N \cup \{v_0\}$, the coefficients that weight each marginal value in our approach are given by the formula:

$$C(i, j) = \begin{cases} \displaystyle\sum_{r=0}^{N(i,j)} \binom{N(i, j)}{r} \frac{(r + 2)!(n - (r + 2) - 1)!}{n!} & \text{if } j \neq 0 \\ \displaystyle\sum_{r=0}^{N(i,0)} \binom{N(i, 0)}{r} \frac{(r + 1)!(n - (r + 1) - 1)!}{n!} & \text{if } j = 0. \end{cases} \tag{2}$$

Summarizing, the Shapley value of a given player $v_k \in N$ is:

$$\phi_k = \sum_{\{v_i, v_j\} \in A_k} C(i, j)(\max(d(v_k, v_i), d(v_k, v_j)) - d(v_i, v_j)) + \sum_{v_i \in B_k} C(i, 0)(\max(d(v_k, v_0), d(v_k, v_i)) - d(v_i, v_0)).$$

For each pair $\{v_i, v_j\}$, $C(i, j)$ can be calculated in $O(n)$ time. Hence, for each $k = 1, \ldots, n$, ϕ_k can be computed in $O(n^2)$ time. Therefore, the complexity of the algorithm to compute the Shapley value is $O(n^3)$.

4.2. Core representation

We have proved above that testing membership in the cores $C(N, v_I)$ and $C(N, v_I^*)$ is NP-hard. Hence, it is very unlikely that these cores have compact representations for general graphs. We will next give a compact representation of the core of these games involving $O(n^2)$ constraints, for tree graphs.

First we note that in this case, if $N = \{v_1, \ldots, v_n\}$, $v_I(N)$ is equal to the diameter of the tree T, [11,12] and can be found by solving the continuous (or absolute) 1-center problem on T, in $O(n)$ time.

More generally, when $N \subseteq V \setminus \{v_0\}$, then for each coalition $S \subseteq N$, $v_I(S)$ is defined as the diameter length of a minimal spanning tree of $S \cup \{v_0\}$. Such a tree, say $T^*(S)$, solves the continuous 1-center problem for the subset of nodes $S \cup \{v_0\}$. Recall that the continuous 1-center problem for some subset $V' \subseteq V$, defines the smallest radius neighborhood in the metric space $A(T)$, which covers V'.

Moreover, $T^*(S)$ has the following property. There is an edge of T, say (v_i, v_j), such that the 1-center of $T^*(S)$ is on this edge, and

$$v_I(S) = d(v_p, v_i) + l(v_i, v_j) + d(v_j, v_q),$$

for some nodes $v_p, v_q \in S \cup \{v_0\}$.

Clearly, the total number of centers of relevant minimum diameter spanning subtrees is $O(n^2)$. In this case each pair of nodes, $v_p, v_q \in N \cup \{v_0\}$ contributes one candidate, denoted by $c_{p,q}$, the midpoint of the unique simple path connecting v_p with v_q. If $d(v_0, c_{p,q}) \leq d(v_p, v_q)/2$, the respective maximal coalition is then defined by

$$S_{p,q} = \{u \in N : d(u, c_{p,q}) \leq d(v_p, v_q)/2\}.$$

If $d(v_0, c_{p,q}) > d(v_p, v_q)/2$, set $S_{p,q} = \emptyset$.

It is then clear that the core of this game is defined by the $O(|N|^2)$ constraints given in the next lemma.

Lemma 4.2. *For a tree graph $T = (V, E)$,*

$$C(N, v_I) = \{x \in \mathbb{R}^N_+ : x(N) = v_I(N),\ x(S_{p,q}) \leq v_I(S_{p,q}), \forall\, p, q \in N \cup \{v_0\}\}.$$

The above polynomial representation of the core implies that membership in the core can be tested in strongly polynomial time by the algorithm in [32].

Remark 4.1. When the tree network is a path, the minimum diameter game coincides with the minimum spanning tree game discussed in [19]. Hence, the efficient algorithms in [19] can be used to efficiently compute, both the nucleolus and the Shapley value.

Conclusions

To summarize, we have shown that $C(N, v_I)$ is always nonempty. Also, $C(N, v_I) \subseteq C(N, v_I^*)$ when $V = N \cup \{v_0\}$. On the other hand, $C(N, v_I^*)$ can be empty if N is a proper subset of $V \setminus \{v_0\}$. Generally, we have proved that for any coalition S, $v_I(S) \leq v_I^*(S) \leq 2v_I(S)$, which in turn implies that any core allocation of $C(N, v_I)$ is also a (1/2)-budget balanced allocation of the game (N, v_I^*).

We have also proved that recognizing whether a given vector x is in the core of the games (N, v_I) and (N, v_I^*) is NP-hard. For tree graphs the games (N, v_I) and (N, v_I^*) coincide and they are submodular. Also for the tree graph case, we have presented a compact formulation of the core, and given a polynomial algorithm to compute the Shapley value.

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